FOURIER SPECTRAL METHODS FOR STOCHASTIC SPACE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY SPECIAL ADDITIVE NOISES

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Abstract. Fourier spectral methods for solving stochastic space fractional partial differential equations driven by special additive noises in one-dimensional case are introduced and analyzed. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. The space-time noise is approximated by the piecewise constant functions in the time direction and by some appropriate approximations in the space direction. The approximated stochastic space fractional partial differential equations are then solved by using Fourier spectral methods. For the linear problem, we obtain the precise error estimates in the L_2 norm and find the relations between the error bounds and the fractional powers. For the nonlinear problem, we introduce the numerical algorithms and MATLAB codes based on the FFT transforms. Our numerical algorithms can be adapted easily to solve other stochastic space fractional partial differential equations with multiplicative noises. Numerical examples for the semilinear stochastic space fractional partial differential equations are given.

Key words. Space fractional partial differential equations, stochastic partial differential equations, Fourier spectral method, error estimates

AMS subject classifications. 65M12; 65M06; Secondary 65M70;35S10

1. Introduction. Fourier spectral methods for solving the following stochastic space fractional partial differential equation are considered in this work, with $1/2 < \alpha \leq 1$,

$$\frac{du(t)}{dt} + A^{\alpha} u(t) = f(u(t)) + \frac{dW(t)}{dt}, \quad 0 < t < T, \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

Here A is an unbounded positive self-adjoint operator, u_0 is an initial value and $f(u)$ is a nonlinear term. The space-time white noise $W(t)$ will be defined below.

Let H be a separable Hilbert space and $\| \cdot \|$, $(\cdot, \cdot)$ denote the norm and inner product in H. Let $A : \mathcal{D}(A) \subset H \to H$ be a positive definite self-adjoint operator such that A^{-1} is compact on H. From this we infer the existence of a complete orthonormal basis $\{e_k\}_{k \geq 0}$ for H of eigenfunctions of A such that the associated sequence of eigenvalues $\{\lambda_k\}$ form an increasing unbounded sequence.

Using the basis $\{e_k\}$ we may also define the fractional powers of A. Given $1/2 < \alpha \leq 1$ define

$$H^{2\alpha} := \mathcal{D}(A^{\alpha}) = \{ v \in H : \sum_k \lambda_k^{2\alpha} |(v, e_k)|^2 < \infty \},$$

and

$$A^{\alpha} v := \sum_k \lambda_k^{\alpha} (v, e_k) e_k, \; v \in \mathcal{D}(A^{\alpha}), \tag{1.3}$$

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with the associated Hilbert norm defined by

$$\|A^{\alpha}v\|^2 = \sum_k \lambda_k^{2\alpha}|(v,e_k)|^2.$$

The special space-time noise considered in this work is

$$\frac{dW(t)}{dt} = \sum_{k=1}^{\infty} \sigma_k(t)\dot{\beta}_k(t)e_k, \tag{1.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k = 1, 2, \dots$ is the derivative of the standard Brownian motions $\beta_k(t)$, $k = 1, 2, \dots$ and $\sigma_k(t), k = 1, 2, \dots$ are some appropriate functions of t. In particular, when $\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0$, the noise (1.4) reduces to

$$\frac{dW(t)}{dt} = \sum_{k=1}^{\infty} \bar{\gamma}_k^{1/2}\dot{\beta}_k(t)e_k,$$

which is a so-called H-valued Wiener process with the covariance operator Q and the linear operator $Q : H \to H$ is a trace class operator, that is $\mathrm{Tr}(Q) = \sum_{k=1}^{\infty} \bar{\gamma}_k < \infty$ where $Qe_k = \bar{\gamma}_k e_k, k = 1, 2, \dots$.

Let us here give two possible operators in (1.1)-(1.2). One is $A = -\Delta$ with the homogeneous Dirichlet boundary condition, $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$, where $\Delta = \partial^2/\partial x^2$ denotes the Laplacian. In this case, A has the eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $e_k = \sqrt{2}\sin k\pi x, k = 1, 2, \dots$. Our error estimates in this work are based on these eigenvalues and eigenfunctions. Another one is $A = I - \Delta$ with periodic boundary conditions, $\mathcal{D}(A) = H^2_{per}(-\pi,\pi)$. Here $H^2_{per}(-\pi,\pi)$ denotes the completion with respect to the $H^2(-\pi,\pi)$ norm of the set of $u \in C^{\infty}([-\pi,\pi])$ such that the pth derivative $u^{(p)}(-\pi) = u^{(p)}(\pi)$ for $p = 0, 1, \dots$. It is a Hilbert space with the $H^2(-\pi,\pi)$ inner product, see [24, Definition 1.47]. In this case, A has the eigenvalues $\lambda_1 = 1, \lambda_{2k} = 1 + k^2, \lambda_{2k+1} = 1 + k^2$ and eigenfunctions $e_1(x) = \frac{1}{\sqrt{2\pi}}, e_{2k}(x) = \frac{1}{\sqrt{\pi}}\sin kx, e_{2k+1}(x) = \frac{1}{\sqrt{\pi}}\cos kx, k = 1, 2, \dots$, see [24, Example 1.84].

We obtain the detailed error estimates, i.e., Theorems 2.1, 3.1, 3.3 below for the linear stochastic space fractional partial differential equation subject to the Dirichlet boundary conditions. More precisely, we shall consider the error estimates for the following linear problem, with $1/2 < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} + (-\Delta)^{\alpha}u(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad 0 < t < T,\ 0 < x < 1, \tag{1.5}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{1.6}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1. \tag{1.7}$$

Here the space-time noise $\frac{\partial^2 W(t,x)}{\partial t \partial x} = \frac{dW(t)}{dt}$ is define by (1.4).

For the linear stochastic space fractional partial differential equation subject to the periodic boundary conditions, we may obtain the similar error estimates as in Theorems 2.1, 3.1, 3.3. For the length of the paper, we will not give the detailed proofs for the error estimates in this case. However, in the numerical examples in Section 4, we shall consider the spectral method for the semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions to illustrate the experimentally determined convergence orders.

The stochastic partial differential equations driven by the white noise (the covariance operator $Q = I$) often have poor regularity estimates. In the physical world, to take into account the short and long range correlations of the stochastic effects, both white noise and colored noises may be considered. There are many situations where colored noises model the reality more closely, and there are also instances where the important stochastic effects are the noises acting on a few selected frequencies. For example one may choose $\sigma_k(t) = \frac{\cos t}{k^3}$. [12]

Space-fractional partial differential equations are widely used to model complex phenomena, for example, quasi-geostrophic flows, fast rotating fluids, dynamic of the frontogenesis in meteorology, diffusion in fractal or disordered medium, pollution problems, mathematical finance and the transport problems, see, e.g., [3], [7], [21], [36], [2].

Let us here consider two examples which apply the fractional Laplacian in the physical models. The first example is about the surface quasi-geostrophic (SQG) equation,

$$\partial_t \theta + \vec{u} \cdot \nabla \theta + \kappa (-\Delta)^{\alpha} \theta = 0,$$

where $\kappa \geq 0$ and $\alpha > 0$, $\theta = \theta(x_1, x_2, t)$ denotes the potential temperature, $\vec{u} = (u_1, u_2)$ is the velocity field determined by θ. When $\kappa > 0$, the SQG equation takes into account the dissipation generated by a fractional Laplacian. The SQG equation with $\kappa > 0$ and $\alpha = 1/2$ arises in geophysical studies of strongly rotating fluids. For the dissipative SQG equation, $\alpha = 1/2$ appears to be a critical index. In the subcritical case when $\alpha > 1/2$, the dissipation is sufficient to control the nonlinearity and the global regularity is a consequence of global a priori bound. In the critical case when $\alpha = 1/2$, the global regularity issue is more delicate. The mystery in the supercritical case $\alpha < 1/2$ is only partially uncovered at the moment. [9]

The second example is about the wave propagation in complex solids, especially viscoelastic materials (for example Polymers).[4]. In this case, the relaxation function has the form $k(t) = ct^{-\nu}, 0 < \nu < 1,\ c \in \mathbb{R}$, instead of the exponential form known in the standard models. This polynomial relaxation is due to the non uniformity of the material. The far field is then described by a Burgers equation with the leading operator $(-\Delta)^{\frac{1+\nu}{2}}$ instead of the Laplacian

$$\partial_t u = -(-\Delta)^{\frac{1+\nu}{2}} u + \partial_x (u^2).$$

This equation also describes the far-field evolution of acoustic waves propagating in a gas-filled tube with a boundary layer.

Frequently, the initial value or the coefficients of the equation are random, therefore it is natural to consider the stochastic space-fractional partial differential equations. The existence, uniqueness and regularities of the solutions of stochastic space-fractional partial differential equations have been extensively studied, see, e.g., [3], [7], [10], [28]. In this work, we will focus on the case $1/2 < \alpha \leq 1$ since the existence and uniqueness and regularity of the solution in this case is well understood in literature, see [11, Theorem 1.3]. However the numerical methods for solving space-fractional stochastic partial differential equations are quite restricted even for the case $1/2 < \alpha \leq 1$. Debbi and Dozzi [11] introduced a discretization of the fractional Laplacian and used it to elaborate an approximation scheme for fractional heat equation perturbed by a multiplicative cylindrical white noise. As far as we know [11] is the only existing paper in the literature of dealing with the numerical approach of this kind of problems. In this work, we will use the ideas developed in [1] and [12], to consider the numerical methods for solving stochastic space fractional partial differential equations, see also [19], [8], [20]. We first approximate the noise by using piecewise constant functions and then obtain the approximate solution $\hat{u}(t)$ of the exact solution $u(t)$. Finally we provide error estimates in L^2-norm for $u(t) - \hat{u}(t)$.

For the deterministic space fractional partial differential equations, many numerical methods are available in literature. There are two approaches to define the fractional Laplacian. One approach is by using the eigenvalues and eigenfunctions of the Laplacian $-\Delta$ subject to the boundary conditions as in (1.3). Another approach is by using the left-handed and right-handed Riemann-Liouville fractional derivatives. For the deterministic space fractional partial differential equations defined by the Riemann-Liouville fractional derivatives, many numerical methods are available, e.g., finite difference methods [14]-[15], [26], [31]-[32], finite element methods [13], [18] and the spectral methods [22]-[23]. For the deterministic space fractional partial differential equations defined by (1.3), some numerical methods are also available, see, e.g., matric transfer technique (MTT) [14], [15], [6], Fourier spectral method [5]. In this work, we will use Fourier spectral method to solve the stochastic space fractional partial differential equations. The main advantage of this approach is that it gives a full diagonal representation of the fractional operator, being able to achieve spectral convergence regardless of the fractional power in the problem.

Let $N_t \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_t} = T$ be the time partition of $[0, T]$ and Δt the time step size. To find the approximate solution of (1.5)-(1.7), we approximate the noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ by the piecewise constant functions in the time direction defined by, with $l = 1, 2, ..., N_t$, [12]

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \sigma_k^M(t) e_k(x) \Big(\sum_{l=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{k,l} \chi_l(t) \Big), \tag{1.8}$$

where

$$\eta_{k,l} = \frac{1}{\sqrt{\Delta t}} \int_{t_{l-1}}^{t_l} d\,\beta_k(t) = \frac{1}{\sqrt{\Delta t}} \Big(\beta_k(t_l) - \beta_k(t_{l-1}) \Big) \in \mathcal{N}(0,1), \tag{1.9}$$

and

$$\chi_l(t) = \begin{cases} 1, & t \in [t_{l-1}, t_l],\ l = 1, 2, \ldots, N_t, \\ 0, & \text{otherwise}. \end{cases}$$

Here $\sigma_k^M(t)$ is the approximation of $\sigma_k(t)$ in the space direction. For example, we can choose, with some positive integer $M > 0$,

$$\sigma_k(t) = \frac{\cos t}{k^3}, \qquad \sigma_k^M(t) = \begin{cases} \sigma_k(t), & k \le M, \\ 0, & k > M. \end{cases}$$

More precisely, replacing $\sigma_k(t)$ by $\sigma_k^M(t)$, we get the noise approximation in space, and replacing $\dot{\beta}_k(t)$ by $\sum_{j=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{k,j} \chi_j(t)$, we get the noise approximation in time.

Substituting $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ with $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ in (1.5)-(1.7), we get

$$\frac{\partial \hat{u}(t,x)}{\partial t} + (-\Delta)^{\alpha} \hat{u}(t,x) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}, \quad 0 < t < T,\ 0 < x < 1, \tag{1.10}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = 0, \quad 0 < t < T, \tag{1.11}$$

$$\hat{u}(0,x) = u_0(x), \quad 0 < x < 1. \tag{1.12}$$

Note that $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ now is a function in $L^2((0,T)\times(0,1))$ and therefore we can solve (1.10)-(1.12) by using any numerical methods for deterministic space fractional partial differential equations. Assume that $\{\sigma_k(t)\}$ and its derivative are uniformly bounded, [12]

$$|\sigma_k(t)| \leq \beta_k, \quad |\sigma_k'(t)| \leq \gamma_k, \quad \forall\, t \in [0,T], \tag{1.13}$$

and the coefficients $\{\sigma_k^M\}$ are constructed such that

$$|\sigma_k(t) - \sigma_k^M(t)| \leq \alpha_k^M, \quad |\sigma_k^M(t)| \leq \beta_k^M, \quad |(\sigma_k^M)'(t)| \leq \gamma_k^M, \quad \forall\, t \in [0,T], \tag{1.14}$$

with positive sequences $\{\alpha_k^M\}$ being arbitrarily chosen, $\{\beta_k^M\}$ and $\{\gamma_k^M\}$ being related to $\{\beta_k\}$ and $\{\gamma_k\}$. Further we assume that

$$\beta_k^M \leq k^{-\tilde{\alpha}}, \quad \text{for some } 0 \leq \tilde{\alpha} < 1/2. \tag{1.15}$$

Let $\mathbb{E}$ denote the expectation, in Theorem 2.1, we prove that, with $1/2 < \alpha \leq 1$ and $0 \leq \tilde{\alpha} < 1/2$,

$$\begin{aligned} &\mathbb{E} \int_0^T \int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2 \, dx dt \\ &\leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^{\alpha}} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^{\alpha} \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1 + \frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big). \end{aligned} \tag{1.16}$$

Let $J \in \mathbb{N}$, we denote

$$S_J = \operatorname{span}\{e_1, e_2, \dots, e_J\},$$

and define by $P_J : H \to S_J$ the projection from H to S_J,

$$P_J v = \sum_{j=1}^{J} (v, e_j) e_j. \tag{1.17}$$

The Fourier spectral method of (1.10)-(1.12) is to find $\hat{u}_J(t) \in S_J$ such that, with $\hat{g}(t,x) := \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$.

$$\frac{\partial \hat{u}_J(t,x)}{\partial t} + (-\Delta)^{\alpha} \hat{u}_J(t,x) = P_J \hat{g}(t,x), \quad 0 < t < T,\ 0 < x < 1, \tag{1.18}$$

$$\hat{u}_J(t,0) = \hat{u}_J(t,1) = 0, \quad 0 < t < T, \tag{1.19}$$

$$\hat{u}_J(0,x) = P_J u_0(x), \quad 0 < x < 1, \tag{1.20}$$

In Theorem 3.1, we prove that, with $1/2 < \alpha \leq 1$ and $0 \leq \tilde{\alpha} < 1/2$,

$$\|\hat{u}(t) - \hat{u}_J(t)\|^2 \leq C \|u_0 - P_J u_0\|^2 + C \frac{1}{(J+1)^{2\alpha}} \int_0^t \|\hat{g}(s)\|^2 \, ds. \tag{1.21}$$

Combining Theorem 2.1 with Theorem 3.1, we have, with $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \leq 1$,

$$
\begin{aligned}
&\mathbb{E} \int_0^T \int_0^1 \big(u(t,x) - \hat{u}_J(t,x)\big)^2 \, dx dt \\
&\leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big) \\
&+ C\mathbb{E}\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}} \Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^{\infty} \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^{\infty} (\beta_k^M)^2 \Big).
\end{aligned}
$$

The paper is organized as follows. In Section 2, we consider the approximation of noise. In Section 3, we introduce the Fourier spectral methods for solving the approximated space fractional partial differential equations and the error estimates for the linear stochastic space fractional partial differential equations are proved. In Section 4, we consider the numerical examples for solving the semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions. From now on we denote by C a generic constant, which may not be the same at different occurrences.

2. Approximate the noise and regularity. It is well known that the mild solution of (1.5)-(1.7) has the following form

$$
u(t,x) = \int_0^1 G_\alpha(t,x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)\, dW(s,y), \tag{2.1}
$$

where

$$
G_\alpha(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j^\alpha t} e_j(x) e_j(y),
$$

and the stochastic integral $\int_0^t \int_0^1 G_\alpha(t-s,x,y)\, dW(s,y)$ is well-defined. The existence and uniqueness of the solutions of (1.5)-(1.7) are discussed in, e.g., [10], [11], [28] and the references cited therein.

Similarly the mild solution of (1.10)-(1.12) has the form of, see, e.g., [12]

$$
\hat{u}(t,x) = \int_0^1 G_\alpha(t,x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)\, d\widehat{W}(s,y), \tag{2.2}
$$

Theorem 2.1. *Let u and $\hat{u}$ be the solutions of* (1.5)-(1.7) *and* (1.10)-(1.12)*, respectively. Assume that the assumptions* (1.13)-(1.15) *hold. Then we have*

$$
\begin{aligned}
&\mathbb{E} \int_0^T \int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2 \, dx dt \\
&\leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big).
\end{aligned} \tag{2.3}
$$

Proof. See the Appendix. □

Remark 2.2. *When $\alpha = 1$, Theorem 2.1 should reduce to the Theorem 3.3 in [12]. However one term $\Delta t^{1/2+\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1/2$ of the bounds in (3.20) in Theorem 3.3 [12] is missing. The term $\Delta t^{1/2+\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1/2$ comes from the estimates II_1 and*

II_3 of the estimate for $II = \mathbb{E}\int_0^T \int_0^1 F_2(t,x)\,dxdt$ in (4.12). The authors in [12] only considered the estimate II_2 and neglected the terms II_1 and II_3 which would produce the term $\Delta t^{1/2+\tilde{\alpha}}, 0 \le \tilde{\alpha} < 1/2$. (See the estimates for the term II in [12, p.1441]). In Theorem 2.1, we include the terms $O(\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}})$.

THEOREM 2.3. *Let $\hat{u}$ be the solution of (1.10)-(1.12). Assume that the assumptions (1.13)-(1.15) hold. Further assume that $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \le 1$ and $\mathbb{E}\|A^\alpha u_0\|^2 < \infty$. Then*

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \Big|\frac{\partial \hat{u}(t,x)}{\partial t}\Big|^2 dxdt \le C\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^\infty \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big), \tag{2.4}$$

and

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \big|A^\alpha \hat{u}(t,x)\big|^2 dxdt \le C\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^\infty \lambda_k^\alpha (\beta_k^M)^2\Big). \tag{2.5}$$

Proof. Assume that, with $0 < t \le t_{j+1}$,

$$\hat{u}(t,x) = \sum_{k=1}^\infty \hat{u}_k(t) e_k(x), \tag{2.6}$$

and, with $\hat{u}_k(0) = (u_0, e_k)$, $k = 1, 2, \dots$,

$$\hat{u}(0,x) = u_0(x) = \sum_{k=1}^\infty \hat{u}_k(0) e_k(x).$$

Substituting (2.6) into (1.10), we get, with $0 < t \le t_{j+1}$,

$$\frac{d\hat{u}_k(t)}{dt} + \lambda_k^\alpha \hat{u}_k(t) = \sigma_k^M(t)\Big(\sum_{l=1}^{j+1} \frac{1}{\sqrt{\Delta t}} \eta_{k,l}\chi_l(t)\Big), \tag{2.7}$$

which implies that, with $0 < t \le t_{j+1}$,

$$\hat{u}_k(t) = e^{-\lambda_k^\alpha t}\hat{u}_k(0) + \int_0^t e^{-\lambda_k^\alpha (t-s)} \sigma_k^M(s)\Big(\sum_{l=1}^{j+1} \frac{1}{\sqrt{\Delta t}} \eta_{k,l}\chi_l(s)\Big)\,ds. \tag{2.8}$$

Let us first show (2.4). Note that $\{e_k\}$ is an orthonormal basis in $H = L^2(0,1)$, we have, by (2.7),

$$\begin{aligned}
&\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \Big|\frac{\partial \hat{u}(t,x)}{\partial t}\Big|^2 dxdt = \mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}} \Big|\frac{d\hat{u}_k(t)}{dt}\Big|^2 dt \\
&\le 2\mathbb{E}\sum_{k=1}^\infty \Big(\int_{t_j}^{t_{j+1}} |\lambda_k^\alpha \hat{u}_k(t)|^2 dt + \int_{t_j}^{t_{j+1}} \Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}} \sum_{l=1}^{j+1} \eta_{k,l}\chi_l(t)\Big|^2 dt\Big) \\
&= 2\mathbb{E}\sum_{k=1}^\infty \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} |\hat{u}_k(t)|^2 dt + 2\mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}} \Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}} \eta_{k,j+1}\chi_{j+1}(t)\Big|^2 dt \\
&= 2(I + II).
\end{aligned}$$

For I, we have, by (2.8), with $t_l^* = t_l, 1 \le l \le j$ and $t_l^* = t,\ l = j+1$,

(2.9)
$$\begin{aligned}
I &\le 2\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\Big|e^{-\lambda_k^\alpha t}\hat{u}_k(0)\Big|^2\,dt + 2\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\Big|\sum_{l=1}^{j+1}\frac{\eta_{k,l}}{\sqrt{\Delta t}}\int_{t_{l-1}}^{t_l^*}e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)\,ds\Big|^2\,dt\\
&= 2\mathbb{E}\sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}e^{-2\lambda_k^\alpha t}\big(A^\alpha u_0, e_k\big)^2\,dt + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j+1}\frac{1}{\Delta t}\Big(\int_{t_{l-1}}^{t_l^*}e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)\,ds\Big)^2\,dt\\
&\le 2\mathbb{E}\sum_{k=1}^{\infty}\big(A^\alpha u_0, e_k\big)^2\Delta t + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j+1}\frac{1}{\Delta t}\Big(\int_{t_{l-1}}^{t_l^*}e^{-2\lambda_k^\alpha(t-s)}\big(\sigma_k^M(s)\big)^2\,ds\Big)\Big(\int_{t_{l-1}}^{t_l^*}1^2\,ds\Big)\,dt\\
&\le 2\mathbb{E}\sum_{k=1}^{\infty}\big(A^\alpha u_0, e_k\big)^2\Delta t + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\Big(\int_0^t e^{-2\lambda_k^\alpha(t-s)}\big(\sigma_k^M(s)\big)^2\,ds\Big)\,dt\\
&\le 2\mathbb{E}\sum_{k=1}^{\infty}\big(A^\alpha u_0, e_k\big)^2\Delta t + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\big(\beta_k^M\big)^2\int_{t_j}^{t_{j+1}}\frac{1-e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha}\,dt\\
&\le 2\mathbb{E}\|A^\alpha u_0\|^2\Delta t + \Delta t\sum_{k=1}^{\infty}\lambda_k^\alpha\big(\beta_k^M\big)^2,
\end{aligned}$$

where in the last inequality, we use the fact $1 - e^{-2\lambda_k^\alpha t} \le 1$.

For II, we have

$$II = \mathbb{E}\sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}\Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\eta_{k,j+1}\chi_{j+1}(t)\Big|^2\,dt = \sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}\Big(\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\Big)^2\,dt \le \sum_{k=1}^{\infty}\big(\beta_k^M\big)^2.$$

Combining I with II we get (2.4). Similarly we have,

$$\begin{aligned}
&\mathbb{E}\int_{t_j}^{t_{j+1}}\int_0^1|A^\alpha\hat{u}(t)|^2\,dxdt = \mathbb{E}\int_{t_j}^{t_{j+1}}\|A^\alpha\hat{u}(t,x)\|^2\,dt\\
&= \mathbb{E}\int_{t_j}^{t_{j+1}}\Big(\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\hat{u}_k^2(t)\Big)\,dt = \mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}|\hat{u}_k(t)|^2\,dt = I,
\end{aligned}$$

which implies (2.5) also holds. Together these estimates complete the proof of Theorem 2.3.
□

3. Fourier spectral method. Denote $E_\alpha(t) = e^{-tA^\alpha}, 1/2 < \alpha \le 1$, where A^α is defined by (1.3). The mild solution of (1.10)-(1.12) has the form of, with $\hat{g}(t) = \frac{\partial^2\widehat{W}(t,x)}{\partial t\partial x}$,

$$\hat{u}(t) = E_\alpha(t)\hat{u}_0 + \int_0^t E_\alpha(t-s)\hat{g}(s)\,ds, \quad \hat{u}(0) = u_0. \tag{3.1}$$

Similarly the solution of (1.18)-(1.20) has the form of

$$\hat{u}_J(t) = E_\alpha(t)P_J\hat{u}_0 + \int_0^t E_\alpha(t-s)P_J\hat{g}(s)\,ds, \quad \hat{u}(0) = P_Ju_0. \tag{3.2}$$

THEOREM 3.1. *Assume that $\hat{u}$ and $\hat{u}_J$ are the solutions of* (1.10)-(1.12) *and* (1.18)-(1.20)*, respectively. Let $0 \le r < 1/2$ and let $u_0 \in H$. Then we have*

$$\big\|A^{r/2}\big(\hat{u}(t) - \hat{u}_J(t)\big)\big\|^2 \le C\big\|u_0 - P_J u_0\big\|^2 + C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}\int_0^t \|\hat{g}(s)\|^2\, ds. \tag{3.3}$$

In particular, with $r = 0$,

$$\|\hat{u}(t) - \hat{u}_J(t)\|^2 \le C\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}}\int_0^t \|\hat{g}(s)\|^2\, ds. \tag{3.4}$$

To prove Theorem 3.1, we need the following smoothing property for the solution operator $E_\alpha(t)$.

LEMMA 3.2.

1. *Let $s > 0$. We have, with $1/2 < \alpha \le 1$,*

$$\|A^s E_\alpha(t)\| \le C t^{-\frac{s}{\alpha}} e^{-\delta t},\ t > 0,$$

for some constants $C = C(s, \alpha) > 0$ and $\delta = \delta(\alpha) > 0$.

2. *Let $P_J : H \to S_J$ be defined by* (1.17)*. We have*

$$\|E_\alpha(t)(I - P_J)v\| \le e^{-t\lambda_{J+1}^\alpha}\|v\|,\ t > 0.$$

Proof. Recall that A is positive definite and A has the eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$. For any function $h(\cdot)$, we have

$$\|h(A)\| = \sup_{\lambda \in \sigma(A)} |h(\lambda)|,$$

where $\sigma(A)$ denotes the set of eigenvalues of A. Thus, with $\delta = \frac{1}{2}\lambda_1^\alpha$,

$$\begin{aligned}
&\|A^s E_\alpha(t)\| = \|A^s E_\alpha(t/2) E_\alpha(t/2)\| \le \|A^s E_\alpha(t/2)\|\|E_\alpha(t/2)\| \\
&= \sup_{\lambda\in\sigma(A)}\big(\lambda^s e^{-\frac{t}{2}\lambda^\alpha}\big)\cdot \sup_{\lambda\in\sigma(A)}\big(e^{-\frac{t}{2}\lambda^\alpha}\big) = \sup_{\lambda\in\sigma(A)}\Big(\frac{\big(\frac{t}{2}\lambda^\alpha\big)^{s/\alpha}}{e^{\frac{t}{2}\lambda^\alpha}}\Big(\frac{t}{2}\Big)^{-s/\alpha}\Big)e^{-\frac{t}{2}\lambda_1^\alpha} \\
&\le C(t/2)^{-s/\alpha}e^{-\delta t} \le C t^{-s/\alpha} e^{-\delta t},
\end{aligned}$$

which shows (1). Further (2) follows from

$$\|E_\alpha(t)(I - P_J)v\| = \Big(\sum_{j=J+1}^\infty e^{-2t\lambda_j^\alpha}(v, e_j)^2\Big)^{1/2} \le e^{-t\lambda_{J+1}^\alpha}\|v\|.$$

Together these estimates complete the proof of Lemma 3.2.
∎

Proof. [Proof of Theorem 3.1] Subtracting (3.2) from (3.1), we get

$$\hat{u}(t) - \hat{u}_J(t) = E_\alpha(t)(u_0 - P_J u_0) + \int_0^t E_\alpha(t-s)\big(\hat{g}(s) - P_J\hat{g}(s)\big)\, ds = I + II. \tag{3.5}$$

For I, we have, with $0 \le r < 1/2$,

$$\begin{aligned}
\|A^{r/2} I\| &= \|A^{\frac{r}{2}} E_\alpha(t)\big(u_0 - P_J u_0\big)\| \\
&= \Big(\sum_{j=J+1}^\infty e^{-2t\lambda_j^\alpha}\lambda_j^r (u_0, e_j)^2\Big)^{1/2} \le e^{-t\lambda_{J+1}^\alpha}\|A^{r/2}(u_0 - P_J u_0)\|.
\end{aligned}$$

For II, we have, by Lemma 3.2, for some $\gamma \in (0,1)$,

$$\begin{aligned}\|A^{r/2} II\| &= \Big\| \int_0^t A^{r/2} E_\alpha(t-s)\big(I-P_J\big)\hat{g}(s)\,ds \Big\| \\ &= \Big\| \int_0^t \Big[A^{r/2} E_\alpha\big((1-\gamma)(t-s)\big)\Big]\Big[E_\alpha\big(\gamma(t-s)\big)(I-P_J)\Big]\hat{g}(s)\,ds \Big\| \\ &\leq C \int_0^t (t-s)^{-\frac{r}{2\alpha}} e^{-\kappa_\alpha (t-s)} \|\hat{g}(s)\|\,ds,\end{aligned}$$

where $\kappa_\alpha = \delta(1-\gamma) + \lambda_{J+1}^\alpha \gamma$.

By Cauchy- Schwarz inequality, we have

$$\|A^{r/2} II\| \leq C\Big(\int_0^t \big((t-s)^{-\frac{r}{2\alpha}} e^{-\kappa_\alpha (t-s)}\big)^2\,ds\Big)^{1/2} \Big(\int_0^t \|\hat{g}(s)\|^2\,ds\Big)^{1/2}.$$

Note that $r<\alpha$, we have, with $\lambda_{J+1} = (J+1)^2\pi^2$,

$$\begin{aligned}\int_0^t \frac{e^{-2\kappa_\alpha s}}{s^{r/\alpha}}\,ds \leq \int_0^\infty \frac{e^{-2\kappa_\alpha s}}{s^{r/\alpha}}\,ds \leq \frac{\int_0^\infty s^{-r/\alpha} e^{-2s}\,ds}{\kappa_\alpha^{1-r/\alpha}} \leq C\frac{1}{\kappa_\alpha^{1-r/\alpha}} \\ \leq C\frac{1}{(\lambda_{J+1}^\alpha)^{1-r/\alpha}} \leq C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}.\end{aligned}$$

Thus

$$\|A^{r/2} II\| \leq C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}} \Big(\int_0^t \|\hat{g}(s)\|^2\,ds\Big)^{1/2}.$$

Together these estimates complete the proof of Theorem 3.1. □

Combining Theorem 2.1 with Theorem 3.1, we have

THEOREM 3.3. *Let u and $\hat{u}_J$ be the solutions of* (1.5)-(1.7) *and* (1.18)-(1.20), *respectively. Assume that the assumptions* (1.13)-(1.15) *hold. Further assume that* $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \leq 1$ *and* $\mathbb{E}\|A^\alpha u_0\|^2 < \infty$. *Then we have*

$$\begin{aligned}&\mathbb{E} \int_0^T \int_0^1 \big(u(t,x) - \hat{u}_J(t,x)\big)^2\,dxdt \\ &\leq C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}}\Big) \\ &+ C\mathbb{E}\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}}\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^\infty \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big).\end{aligned}$$

Proof. Note that

$$\begin{aligned}&\mathbb{E} \int_0^T \int_0^1 \big(u(t,x) - \hat{u}_J(t,x)\big)^2\,dxdt \\ &\leq 2\mathbb{E} \int_0^T \int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2\,dxdt + 2\mathbb{E} \int_0^T \int_0^1 \big(\hat{u}(t,x) - \hat{u}_J(t,x)\big)^2\,dxdt \\ &= 2I + 2II.\end{aligned}$$

For I, we have, by Theorem 2.1,

$$I \leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big).$$

For II, we have

$$II = \mathbb{E} \int_0^T \|\hat{u}(t) - \hat{u}_J(t)\|^2 \, dt \leq C\mathbb{E}\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}} \mathbb{E} \int_0^T \int_0^t \|\hat{g}(s)\|^2 \, ds dt.$$

Note that $\hat{g}(s) = \frac{d\hat{u}(s)}{ds} + (-\Delta)^\alpha \hat{u}(s)$, we have, by Theorem 2.3,

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_0^t \|\hat{g}(s)\|^2 \, ds dt \leq \mathbb{E} \int_0^T \int_0^t \Big\| \frac{d\hat{u}(s)}{ds} + (-\Delta)^\alpha \hat{u}(s) \Big\|^2 \, ds dt \\
&\leq C\mathbb{E} \int_0^T \int_0^T \int_0^1 \Big(\Big| \frac{\partial \hat{u}(s,x)}{\partial s} \Big|^2 + |(-\Delta)^\alpha \hat{u}(s,x)|^2 \Big) \, dx ds dt \\
&\leq C\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^{\infty} \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^{\infty} (\beta_k^M)^2 \Big).
\end{aligned}$$

Together these estimates complete the proof of Theorem 3.3.
□

4. Numerical simulations. In this section, we will consider the numerical simulation of the Fourier spectral methods for solving the following semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions, with $1/2 < \alpha \leq 1, \ 0 < x < 1, \ 0 < t \leq T$,

$$\frac{\partial u(t,x)}{\partial t} + \epsilon(-\Delta)^\alpha u(t,x) = f(u(t,x)) + \frac{\partial^2 W(t,x)}{\partial t \partial x}, \tag{4.1}$$

$$u(t,0) = u(t,1), \quad u_x'(t,0) = u_x'(t,1), \tag{4.2}$$

$$u(0,x) = u_0(x), \tag{4.3}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian defined by using the eigenvalues and eigenfunctions of the Laplacian $-\Delta$ subject to the periodic boundary conditions. Here $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and $\epsilon > 0$ denotes the diffusion coefficient. Here we consider the problems with the periodic boundary conditions because we want to compare our numerical results with the results in [24, Example 10.39] where the algorithms of the spectral methods for stochastic semilinear parabolic equation subject to the periodic boundary conditions are given and discussed. One may also consider the algorithms and MATLAB codes for stochastic space fractional partial differential equations with the homogeneous boundary conditions following the approaches in, e.g., [16], [17]. Although the Laplacian is singular in (4.1)-(4.2) due to the periodic boundary conditions, we expect the errors to behave as in Theorem 3.3, see the comments in [24, Corollary 10.38].

Denote $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) = H^2_{per}(0,1)$, where $\mathcal{D}(A) = H^2_{per}(0,1)$ is defined in the Introduction section. Then the eigenvalues and eigenfunctions of A can also be expressed by

$$\lambda_k = (2\pi k)^2, \quad e_k = e^{i2\pi kx}, \ k \in \mathbb{Z}.$$

The noise has the form of

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}} \sigma_k(t) \dot{\beta}_k(t) e_k(x), \tag{4.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k \in \mathbb{Z}$ are the derivatives of the standard Brownian motions $\beta_k(t)$, $k \in \mathbb{Z}$ and $\sigma_k(t), k \in \mathbb{Z}$ are some appropriate functions of t. Here $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. When $\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0, k \in \mathbb{Z}$, the noise (4.4) reduces to

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}} \bar{\gamma}_k^{1/2} \dot{\beta}_k(t) e_k(x). \tag{4.5}$$

The approximate noise $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ is, with some positive integer $M > 0$,

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}, |k| \le M} \bar{\gamma}_k^{1/2} e_k(x) \sum_{l=1}^{N_t} \frac{\eta_{k,l}}{\Delta t} \chi_l(t). \tag{4.6}$$

In our numerical example below, we assume that, [24, Example 10.8],

$$\bar{\gamma}_0 = 0, \quad \bar{\gamma}_k = |k|^{-(2r_1+1+\tilde{\epsilon})}, \ k \in \mathbb{Z}, \ k \neq 0. \tag{4.7}$$

where $\tilde{\epsilon} > 0$ is a very small positive number. When $r_1 = -1/2$, we obtain so-called space-time white noise. When $r_1 = 1$, we obtain the smooth noise.

Let $S_J := \text{span}\{e_0, e_1, \dots, e_{J/2}, e_{-J/2+1} \dots, e_{-1}\}$. We assume $J \le M$ where M is determined in (4.5). Here the ordering $0, 1, 2, \dots, J/2, -J/2+1, \dots, -1$ is consistent with the ordering in the MATLAB functions **fft** and **ifft** [33]. Let $0 = t_0 < t_1 < t_2 < \dots < t_{N_t} = T$, $N_t \in \mathbb{N}$ be the time partition of $[0, T]$ and Δt the time step size with $T = N_t \Delta t$. We use the semi-implicit Euler method to consider the time discretization.

We will consider the convergence rate against the different time steps. Choose $J = 64$. The reference solution is obtained by using the time step size $\Delta tref = T/Nref$ with $Nref = 10^4$. Let $\mathbf{kappa} = [5, 10, 20, 50, 100, 200, 500]$, we will consider the approximate solutions with the different time step sizes $\Delta t_i = \Delta tref * \mathbf{kappa}(i), i = 1, 2, \dots, 7$. By Theorem 2.1, we have

$$\begin{aligned} &\mathbb{E} \int_0^T \int_0^1 \Big(u(t,x) - \hat{u}(t,x) \Big)^2 dx dt \\ &\le C \Big(\sum_{k \in \mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k \in \mathbb{Z}} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 + \Delta t^{1 + \frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big). \end{aligned} \tag{4.8}$$

We remark that here we choose $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. In our numerical example, we will choose, with $\bar{\gamma}_k$ given by (4.7),

$$\begin{aligned} &\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0, k \in \mathbb{Z}, \\ &\sigma_k^M(t) = \begin{cases} \sigma_k(t) = \bar{\gamma}_k^{1/2}, \ |k| \le M, \\ 0, \quad |k| > M, \end{cases} \end{aligned}$$

which implies that

$$|\sigma_k^M(t)| \le \beta_k^M, \text{ where } \beta_k^M = \bar{\gamma}_k^{1/2}, \ |k| \le M,$$

and

$$|\sigma_k(t) - \sigma_k^M(t)| \le \alpha_k^M, \text{ where } \alpha_k^M = \bar{\gamma}_k^{1/2}, \ |k| > M.$$

We first observe that for sufficiently large M the convergence order of the L^2 norm of the error in (4.8) is dominated by $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big)$. In fact, we will choose $M = J$ where J is sufficiently large. Then the first term of the right side of (4.8) satisfies, with $\lambda_k = (2\pi k)^2, k \in \mathbb{Z}$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} &= \sum_{|k|>M} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} \le C\Big(\frac{1}{\lambda_{M+1}^\alpha} + \frac{1}{\lambda_{M+2}^\alpha} + \dots\Big) \\ &\le C\Big(\frac{1}{(M+1)^{2\alpha}} + \frac{1}{(M+2)^{2\alpha}} + \dots\Big) \\ &= C\Big(\frac{1}{(J+1)^{2\alpha}} + \frac{1}{(J+2)^{2\alpha}} + \dots\Big).\end{aligned}$$

The second term of the right side of the error in (4.8) is $O(\Delta t^2)$. Hence for sufficiently large J, the convergence order of the L^2 norm of the error in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big)$.

We now consider two cases $r_1 = -1/2$ and $r_1 = 1$ in (4.7). For $r_1 = -1/2$, we may choose $\tilde{\alpha} = 0$ which implies that the convergence order of the L^2 norm in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big) = O\big(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}\big)$. Indeed, $\tilde{\alpha} = 0$ satisfies (1.15), that is,

$$\beta_k^M = \bar{\gamma}_k^{1/2} = |k|^{-\frac{2r_1+1+\bar{\epsilon}}{2}} = |k|^{-\bar{\epsilon}/2} \le |k|^{-\tilde{\alpha}}.$$

For $r_1 = 1$, we may choose $\tilde{\alpha} = 1/2 - \bar{\epsilon}$ (since $0 \le \tilde{\alpha} < 1/2$) with arbitrarily small positive number $\bar{\epsilon}$ which implies that the convergence order of the L^2 norm in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big) = O\big(\Delta t^{\frac{1}{2}(1-\frac{\bar{\epsilon}}{\alpha})}\big) \approx O(\Delta t^{1/2})$. Indeed, in this case, $\tilde{\alpha} = 1/2 - \bar{\epsilon}$ satisfies (1.15), that is,

$$\beta_k^M = \bar{\gamma}_k^{1/2} = |k|^{-\frac{2r_1+1+\bar{\epsilon}}{2}} = |k|^{-\frac{3+\bar{\epsilon}}{2}} \le |k|^{-\tilde{\alpha}}.$$

Thus we have, by Theorem 2.1, the following error estimates, with $1/2 < \alpha \le 1$ and $r_1 = -1/2$,

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \le C(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}), \tag{4.9}$$

and, with $1/2 < \alpha \le 1$ and $r_1 = 1$

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \le C(\Delta t^{1/2}), \tag{4.10}$$

where the norm is measured in L^2 both for time and space. In particular, when $\alpha = 1, r_1 = -1/2$, we have

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \le C(\Delta t^{1/4}),$$

which is consistent with the standard time discretization error for the stochastic heat equation driven by space-time white noise, see, e.g., [35].

In our numerical experiment below, we choose $f(u) = u - u^3$, $u_0(x) = \sin(2\pi x)$, and $\epsilon = 1$. See the simulation of this problem for $\alpha = 1$ in [30]. We will consider the error estimates $\|\hat{u}(t_n) - u(t_n)\|_{L^2(\Omega,H)}$ at time t_n. We hope to observe the same convergence order as in (4.9) and (4.10).

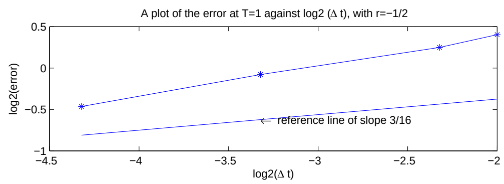

FIG. 1. *A plot of the error at $T = 1$ against log2(Δt) with $\alpha = 0.8, r_1 = -1/2$*

To do this, we consider $\overline{M} = 100$ simulations. For each simulation $\omega_m, m = 1, 2, \dots, \overline{M}$, we generate J independent Brownian motions $\beta_l, l = 0, 1, \dots, J/2, -J/2+1, \dots, -1$ and compute $\hat{u}_J(t_n) \approx \hat{u}(t_n)$ at time $t_n = 1$ by using the different time step sizes. We then compute the following L^2 norm of the error at $t_n = 1$ for the simulation $\omega_m, m = 1, 2, \dots, \overline{M}$,

$$\epsilon(\Delta t_i, \omega_m) = \epsilon(\Delta t_i, \omega_m, t_n) = \|\hat{u}_J(t_n, \omega_m) - \mathrm{uref}(t_n, \omega_m)\|^2,$$

where the reference ("true ") solution $\mathrm{uref}(t_n, \omega_m)$ is approximated by using the time step $\Delta tref = T/Nref$ and $Jref = J$. We then average $\epsilon(\Delta t_i, \omega_m)$ with respect to ω_m to obtain the following approximation of $\|\hat{u}_J(t_n) - \mathrm{uref}(t_n)\|_{L^2(\Omega, H)}$ for the different time step size Δt_i,

$$S(\Delta t_i) = \Big(\frac{1}{\overline{M}} \sum_{m=1}^{\overline{M}} \epsilon(\Delta t_i, \omega_m) \Big)^{1/2} = \Big(\frac{1}{\overline{M}} \sum_{m=1}^{\overline{M}} \|\hat{u}_J(t_n, \omega_m) - \mathrm{uref}(t_n, \omega_m)\|^2 \Big)^{1/2}.$$

For example, in the case $\alpha = 0.8, r_1 = -1/2$, the convergence rate against the time step size is $O(\Delta t^{\frac{1}{2}(1 - \frac{1}{2\alpha})}) = O(\Delta t^{3/16})$, i.e., with some positive constant C,

$$S(\Delta t_i) \approx C \Delta t_i^{3/16},$$

which implies that

$$\log(S(\Delta t_i)) \approx \log(C) + \frac{3}{16} \log(\Delta t_i), i = 1, 2, \dots, 7.$$

In Figure 1, we consider the case $\alpha = 0.8, r_1 = -1/2$ and plot the points $\big(\log(\Delta t_i), \log(S(\Delta t_i))\big), i = 1, 2, \dots, 7$ and we observe that the experimentally determined convergence order is higher than the theoretical order in this case. Here the reference line has the slope $\frac{3}{16}$.

In Figure 2, we consider the case $\alpha = 0.8, r_1 = 1$ and in this case the theoretical convergence order with respect to the time step size is $O(\Delta t^{1/2})$. We plot the points $\big(\log(\Delta t_i), \log(S(\Delta t_i))\big), i = 1, 2, \dots, 7$ and we observe that the experimentally determined convergence order is also higher than the theoretical order in this case. Here the reference line has the slope $1/2$.

In Figure 3, we consider the convergence rate against the different J. Choose fixed time step $\Delta t = T/N_t$ with $N_t = 10^4$. We then consider the different $J = Jref * (\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^8})$ where $Jref = 2^{10}$.

We will first generate the reference Brownian motions

$$\beta_j(t), j = 0, 1, 2, \dots, Jref/2, -Jref/2 + 1, \dots - 1 \tag{4.11}$$

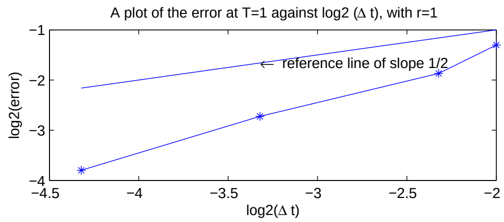

FIG. 2. *A plot of the error at* $T = 1$ *against* $log2(\Delta t)$ *with* $\alpha = 0.8, r_1 = 1$

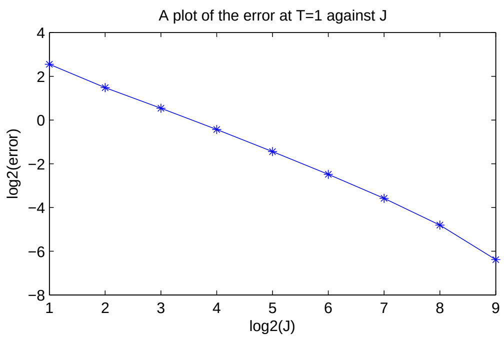

FIG. 3. *A plot of the error at* $T = 1$ *against the* J *with* $\alpha = 0.8, r_1 = 1$

for computing the reference ("true") solution uref. When we consider the approximate solution u with J truncated terms, we will use the Brownian motions $\beta_j(t), j = 0, 1, 2, \dots, J/2, -J/2 + 1, \cdots - 1$ from (4.11).

In Figure 3, we consider the case $\alpha = 0.8, r_1 = 1$ and plot the L^2 norm error against the different J where the L^2 norm error are approximated by using $\overline{M} = 100$ simulations. We indeed observe the spectral convergence with respect to the different J.

Appendix In the Appendix, we shall provide the proof of Theorem 2.1. To do this, we need the following lemma.

LEMMA 4.1. *Let* $1/2 < \alpha \leq 1$ *and* $0 \leq \tilde{\alpha} < 1/2$. *We have*

$$\int_0^\infty x^{-2(\tilde{\alpha}+\alpha)} \big(1 - e^{-x^{2\alpha}\Delta t}\big)\, dx \leq C \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}.$$

Proof. With the variable change $y = x^{2\alpha}\Delta t$, we have

$$\int_0^\infty x^{-2(\tilde{\alpha}+\alpha)} \big(1 - e^{-x^{2\alpha}\Delta t}\big)\, dx = C \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}} \Big(\int_0^1 + \int_1^\infty \Big) \frac{1 - e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy$$

It is easy to see that, with $\alpha \in (1/2, 1]$,

$$\int_1^\infty \frac{1 - e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy \leq C.$$

Further, we have

$$\Big| \int_0^1 \frac{1 - e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy \Big| \leq C \int_0^1 \frac{y}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy \leq C \int_0^1 \frac{1}{y^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy < \infty.$$

if $1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}<1$, i.e., $0\le\tilde{\alpha}<1/2$.

Together these estimates complete the proof of Lemma 4.1.

□

Proof. [Proof of Theorem 2.1]

Subtracting (2.2) from (2.1), we have

$$
\begin{aligned}
&u(t,x)-\hat{u}(t,x)\\
&=\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\\
&=\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]\\
&+\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\Big]\\
&=F_1(t,x)+F_2(t,x),
\end{aligned}
$$

where, with $\eta_{k,l}$ and $\chi_l(t)$ defined as in (1.9),

$$
\begin{aligned}
dW(s,y)&=\frac{\partial^2 W(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^{\infty}\sigma_k(s)e_k(y)\Big]\,d\beta_k(s)dy,\\
d\overline{W}(s,y)&=\frac{\partial^2 \overline{W}(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^{\infty}\sigma_k^M(s)e_k(y)\Big]\,d\beta_k(s)dy,\\
d\widehat{W}(s,y)&=\frac{\partial^2 \widehat{W}(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^{\infty}\sigma_k^M(s)\Big(\sum_{l=1}^{N_t}\frac{\eta_{k,l}}{\sqrt{\Delta t}}\chi_l(s)\Big)e_k(y)\Big]\,dsdy.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbb{E}\int_0^T\int_0^1|u(t,x)-\hat{u}(t,x)|^2\,dxdt&\le C\mathbb{E}\int_0^T\int_0^1 F_1^2(t,x)\,dxdt\\
&+C\mathbb{E}\int_0^T\int_0^1 F_2^2(t,x)\,dxdt=C(I+II).
\end{aligned}
$$

For I, we have, by using isometry property and (1.14), with $G_\alpha(t-s,x,y)=\sum_{j=1}^{\infty}e^{-(t-s)\lambda_j^\alpha}e_j(x)e_j(y)$,

$$
\begin{aligned}
I&=\mathbb{E}\int_0^T\int_0^1\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2\,dxdt\\
&=\int_0^T\int_0^1\int_0^t\Big[\int_0^1 G_\alpha(t-s,x,y)\Big(\sum_{k=1}^{\infty}\big(\sigma_k(s)-\sigma_k^M(s)\big)e_k(y)\Big)\,dy\Big]^2\,dsdxdt.\\
&=\int_0^T\int_0^t\sum_{k=1}^{\infty}e^{-2(t-s)\lambda_k^\alpha}\big(\alpha_k^M\big)^2\,dsdt=\int_0^T\sum_{k=1}^{\infty}\frac{1-e^{-2t\lambda_k^\alpha}}{2\lambda_k^\alpha}\big(\alpha_k^M\big)^2\,dt\le C\sum_{k=1}^{\infty}\frac{1}{2\lambda_k^\alpha}\big(\alpha_k^M\big)^2.
\end{aligned}
$$

For II, we have

$$
\begin{aligned}
II &= \mathbb{E}\int_0^T \int_0^1 \Big\{ \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y)\, d\overline{W}(s,y) - \int_0^t \int_0^1 G_\alpha(t-s,x,y)\, d\widehat{W}(s,y) \Big]^2 \Big\} dxdt \\
&\leq 3\mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big\{ \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y)\, d\overline{W}(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)\, d\overline{W}(s,y) \Big]^2 \\
&+ \Big[\int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)\, d\overline{W}(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)\, d\widehat{W}(s,y) \Big]^2 \\
&+ \Big[\int_0^{t_j} \int_0^1 G_\alpha(t_j-s,x,y)\, d\widehat{W}(s,y) - \int_0^t \int_0^1 G_\alpha(t_j-s,x,y)\, d\widehat{W}(s,y) \Big]^2 \Big\} dxdt \\
&\leq 3\big(II_1 + II_2 + II_3\big).
\end{aligned} \tag{4.12}
$$

For II_2, we have, by isometry property,

$$
\begin{aligned}
II_2 &= \mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \int_0^1 G_\alpha(t_j-s,x,y) \Big(\sum_{k=1}^{\infty} \sigma_k^M(s) e_k(y)\, dy \Big) d\beta_k(s) \\
&- \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \int_0^1 G_\alpha(t_j-\tilde{s},x,y) \sum_{k=1}^{\infty} \sigma_k^M(\tilde{s}) e_k(y)\, dy d\tilde{s} \Big(\frac{1}{\Delta t} \int_{t_l}^{t_{l+1}} d\beta_k(s) \Big) \Big]^2 dxdt \\
&= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \Big\{ \int_0^1 G_\alpha(t_j-s,x,y) \Big(\sum_{k=1}^{\infty} \sigma_k^M(s) e_k(y) \Big) dy \\
&- \frac{1}{\Delta t} \int_{t_l}^{t_{l+1}} \int_0^1 G_\alpha(t_j-\tilde{s},x,y) \Big(\sum_{k=1}^{\infty} \sigma_k^M(\tilde{s}) e_k(y) \Big) dy d\tilde{s} \Big\}^2 dsdxdt \\
&= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \Big\{ \frac{1}{\Delta t} \int_{t_l}^{t_{l+1}} \Big[\int_0^1 G_\alpha(t_j-s,x,y) \Big(\sum_{k=1}^{\infty} \sigma_k^M(s) e_k(y) \Big) dy \\
&- \int_0^1 G_\alpha(t_j-\tilde{s},x,y) \Big(\sum_{k=1}^{\infty} \sigma_k^M(\tilde{s}) e_k(y) \Big) dy \Big] d\tilde{s} \Big\}^2 dsdxdt \\
&= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \sum_{k=1}^{\infty} \Big\{ \frac{1}{\Delta t} \int_{t_l}^{t_{l+1}} \Big[e^{-\lambda_k^\alpha (t_j-s)} \sigma_k^M(s) - e^{-\lambda_k^\alpha (t_j-\tilde{s})} \sigma_k^M(\tilde{s}) \Big] d\tilde{s} \Big\}^2 dsdt \\
&= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2} \Big\{ \int_{t_l}^{t_{l+1}} \Big[e^{\lambda_k^\alpha s} \sigma_k^M(s) - e^{\lambda_k^\alpha \tilde{s}} \sigma_k^M(\tilde{s}) \Big] d\tilde{s} \Big\}^2 dsdt.
\end{aligned}
$$

By (1.14), we have, with some ξ_l^1, ξ_l^2 which lie between s and $\tilde{s}$,

$$
\begin{aligned}
&\Big| e^{\lambda_k^\alpha s} \sigma_k^M(s) - e^{\lambda_k^\alpha \tilde{s}} \sigma_k^M(\tilde{s}) \Big| = \Big| \Big(e^{\lambda_k^\alpha s} - e^{\lambda_k^\alpha \tilde{s}} \Big) \sigma_k^M(s) + e^{\lambda_k^\alpha \tilde{s}} (\sigma_k^M(s) - \sigma_k^M(\tilde{s}) \Big| \\
&\leq \Big| \Big(\lambda_k^\alpha e^{\lambda_k^\alpha \xi_l^1} \Delta t \Big) \sigma_k^M(s) + e^{\lambda_k^\alpha \tilde{s}} \Big((\sigma_k^M)'(\xi_l^2) \Big) \Delta t \Big| \\
&\leq \Big| \lambda_k^\alpha e^{\lambda_k^\alpha t_{l+1}} \beta_k^M \Delta t + e^{\lambda_k^\alpha t_{l+1}} \gamma_k^M \Delta t \Big| \leq e^{\lambda_k^\alpha t_{l+1}} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big) \Delta t.
\end{aligned}
$$

Hence

$$II_2 \leq \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\sum_{k=1}^{\infty}\frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2}\Big[e^{2\lambda_k^\alpha t_{l+1}}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2\Delta t^4\Big]\,dsdt$$
$$\leq \Delta t^2\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^{t_j}\sum_{k=1}^{\infty}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2\,dsdt \leq C\Delta t^2\sum_{k=1}^{\infty}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2,$$

where we use the inequality $e^{-2\lambda_k^\alpha(t_j-t_{l+1})}\leq 1$ for $l=0,1,2,\ldots,j-1$.

For II_1, we have

$$II_1 = \mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)-\int_0^{t_j}\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2\,dxdt$$
$$\leq 2\mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_0^{t_j}\int_0^1\Big(G_\alpha(t-s,x,y)-G_\alpha(t_j-s,x,y)\Big)\,d\overline{W}(s,y)\Big]^2\,dxdt$$
$$+\,2\mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_{t_j}^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2\,dxdt = 2(II_1^1+II_1^2).$$

For II_1^1, we have, by the isometry property and (1.14),

$$II_1^1 = \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^{t_j}\sum_{k=1}^{\infty}\Big(e^{-\lambda_k^\alpha(t-s)}-e^{-\lambda_k^\alpha(t_j-s)}\Big)^2(\sigma_k^M(s))^2\,dsdt$$
$$\leq \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{k=1}^{\infty}(\beta_k^M)^2\int_0^{t_j}\Big(e^{-\lambda_k^\alpha(t-s)}-e^{-\lambda_k^\alpha(t_j-s)}\Big)^2\,dsdt$$

Note that

$$\int_0^{t_j}\Big(e^{-\lambda_k^\alpha(t-s)}-e^{-\lambda_k^\alpha(t_j-s)}\Big)^2\,ds = \int_0^{t_j}e^{-2\lambda_k^\alpha(t-s)}\Big(1-e^{-\lambda_k^\alpha(t_j-t)}\Big)^2\,ds$$
$$=\Big(1-e^{-\lambda_k^\alpha(t_j-t)}\Big)^2\frac{e^{-2\lambda_k^\alpha(t-t_j)}-e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha}\leq\frac{\Big(1-e^{-\lambda_k^\alpha(t-t_j)}\Big)^2}{2\lambda_k^\alpha}.$$

Hence, we have

$$II_1^1\leq\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\Big(\sum_{k=1}^{\infty}(\beta_k^M)^2\Big)\frac{\Big(1-e^{-\lambda_k^\alpha(t-t_j)}\Big)^2}{2\lambda_k^\alpha}\,dt\leq C\sum_{k=1}^{\infty}(\beta_k^M)^2\frac{\Big(1-e^{-\lambda_k^\alpha\Delta t}\Big)^2}{2\lambda_k^\alpha}.$$

By (1.15) and Lemma 4.1, we have

$$II_1^1\leq C\sum_{k=1}^{\infty}k^{-2\tilde\alpha}\frac{\Big(1-e^{-\lambda_k^\alpha\Delta t}\Big)^2}{2\lambda_k^\alpha}\leq C\int_1^\infty x^{-2(\tilde\alpha+\alpha)}\big(1-e^{-x^{2\alpha}\Delta t}\big)\,dx\leq C\Delta t^{1+\frac{\tilde\alpha}{\alpha}-\frac{1}{2\alpha}}.$$

For II_1^2, we have, by isometry property and (1.14) and (1.15),

$$
\begin{aligned}
II_1^2 &= \mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_{t_j}^{t} \int_0^1 G_\alpha(t-s,x,y)\, d\overline{W}(s,y) \Big]^2 dxdt \\
&= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t} \sum_{k=1}^{\infty} e^{-2\lambda_k^\alpha (t-s)} \big(\sigma_k^M(s)\big)^2 dsdt \leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t} \sum_{k=1}^{\infty} \Big(k^{-2\tilde{\alpha}} e^{-2\lambda_k^\alpha (t-s)} \Big) dsdt \\
&\leq C \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{k=1}^{\infty} \Big[k^{-2\tilde{\alpha}} \Big(\frac{1-e^{-2\lambda_k^\alpha \Delta t}}{\lambda_k^\alpha} \Big) \Big] dt = C \sum_{k=1}^{\infty} \Big[k^{-2\tilde{\alpha}} \Big(\frac{1-e^{-2\lambda_k^\alpha \Delta t}}{\lambda_k^\alpha} \Big) \Big] \\
&\leq C \int_0^\infty \frac{1-e^{-2x^{2\alpha}\Delta t}}{x^{2\alpha+2\tilde{\alpha}}}\, dx \leq C \int_0^\infty x^{-2(\tilde{\alpha}+\alpha)} \big(1-e^{-x^{2\alpha}\Delta t}\big)\, dx.
\end{aligned}
$$

By Lemma 4.1, we have

$$II_1^2 \leq C \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}. \tag{4.13}$$

Similarly we may show, with $0 \leq \tilde{\alpha} < 1/2$,

$$II_3 \leq C \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}.$$

Together these estimates complete the proof of Theorem 2.1.

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REFERENCES

[1] E. J. Allen, S. J. Novosel and Z. Zhang, *Finite element and difference approximation of some linear stochastic partial differential equations*, Stochastics Stochastics Rep., 64(1998), pp. 117-142.

[2] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, *The fractional order governing equations of Lévy motion*, Water Resour. Res., 36 (2000), pp. 1413-1423.

[3] P. Biler, T. Funaki, , and W. A. Woyczynski, *Fractal Burgers' equations*, J. Differential Equations, 148 (1998), pp. 9-46.

[4] Z. Brzeźniak, L. Debbi and B. Goldys, *Ergodic properties of fractional stochastic Burgers Equations*, Global and Stochastic Analysis, 1(2011), 149-174.

[5] A. Bueno-Orovio, D. Kay and K. Burrage, *Fourier spectral methods for fractional-in-space reaction-diffusion equations*, BIT Numer. Math., 2014, DOI 10.1007/s10543-014-0484-2.

[6] K. Burrage, N. Hale, and D. Kay, *An efficient implicit FEM scheme for fractional-in-space reaction-diffusion equations*, SIAM J. Sci. Comput., 34 (2012), pp. A2145-A2172.

[7] L. A. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math., 171 (2010), pp. 1903-1930.

[8] Y. Cao, H. Yang and H. Yin, *Finite element methods for semilinear elliptic stochastic partial differential equations*, Numer. Math. 106(2007), 181-198.

[9] P. Constantin, M. C. Lai, R. Sharma, Y. H. Tseng and J. Wu, *New numerical results for the surface quasi-geostrophic equation*, J. Sci. Comput., 50 (2012), 1-28.

[10] L. Debbi and M. Dozzi, *On the solutions of nonlinear stochastic fractional partial differential equations in one spatial dimension*, Stoc. Proc. Appl., 115(2005), pp. 1764-1781.

[11] L. Debbi and M. Dozzi, *On a space discretization scheme for the fractional stochastic heat equations*, arXiv: 1102.4689v1, 2011

[12] Q. Du and T. Zhang, *Numerical approximation of some linear stochastic partial differential equations driven by special additive noises*, SIAM J. Numer. Anal., 40(2002), pp. 1421-1445.

[13] V. J. Ervin and J. P. Roop, *Variational solution of fractional advection dispersion equations on bounded domains in* $\mathbb{R}^d$, Numer. Methods Partial Differential Equations, 23(2007), pp. 256-281.

[14] M. Ilic, F. liu, I. Turner and V. Anh, *Numerical approximation of a fractional-in-space diffusion equation I*, Frac. Calc. Appl. Anal., 8(2005), pp. 323-341.

[15] M. Ilic, F. liu, I. Turner and V. Anh, *Numerical approximation of a fractional-in-space diffusion equation II: with nonhomogeneous boundary conditions*, Frac. Calc. Appl. Anal., 9(2006), pp. 333-349.

[16] A. Jentzen and P. E. Kloeden, *The numerical approximation of stochastic partial differential equations*, Milan J. Math., 77(2009), pp. 205-244.

[17] A. Jentzen and M. Röckner, *A break of the complexity of the numerical approximation of nonlinear SPDEs with multiplicative noise*, Preprint, arXiv:1001.2751v2, 2010.

[18] B. Jin, R. Lazarov, J. Pasciak and Z. Zhou, *Error analysis of a finite element method for the space-fractional parabolic equation*, SIAM J. Numer. Anal. 52(2014), pp. 2272-2294.

[19] M. A. Katsoulakis, G. T. Kossioris and O. Lakkis, *Noise regularization and computations for the 1-dimensional stochastic Allen-Cahn problem*, Interfaces Free Bound. 9 (2007), pp. 1-30.

[20] G. T. Kossioris and G. E. Zouraris, *Fully-discrete finite element approximations for a fourth-order linear stochastic parabolic equation with additive space-time white noise*, M2AN Math. Model. Numer. Anal., 44 (2010), pp. 289-322.

[21] X. Leoncini and G. M. Zaslavsky, *Jets, stickiness and anomalous transport*, Phys. Rev. E, 65(046216), 2002.

[22] X. J. Li and C. J. Xu, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal., 47(2009), pp. 2108-2131

[23] X. J. Li and C. J. Xu , *Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation*, Commun. Comput. Phys., 8(2010), pp. 1016-1051

[24] G. J. Lord, C. E. Powell, and T. Shardlow, *An Introduction to Computational Stochastic PDEs*, Cambridge University Press, Cambridge, UK, 2014.

[25] S. N. Majumdar and A. J. Bray, *Spatial persistence of fluctuating interfaces*, Phys. Rev. Lett., 86(2011), pp. 3700-3703.

[26] M. M. Meerschaert and C. Tadjeran, *Finite difference approximations for fractional advection-dispersion flow equations*, J. Comput. Appl. Math., 172 (2004), pp. 65-77.

[27] J. P. Roop, *Computational aspects of FEM approximation of fractional advection dispersion equations on bouinded domains in* $\mathbb{R}^2$, J. Comput. Appl. Math., 193(2006), pp. 243-268.

[28] M. Röckner, R. C. Zhu and X. C. Zhu, *Local existence and non-explosion of solutions for stochastic fractional partial differential equations driven by multiplicative noise*, Stochastic Processes and their Applications 124(2014), pp. 1974 -2002.

[29] R. Servadei and E. Valdinoci, *On the spectrum of two different fractional operators*, Proc. Royal Soc. Edinburgh Sec. A., 144(2014), pp. 831-855.

[30] T. Shardlow, *Stochastic perturbations of the Allen-Cahn equation*, Electron. J. Differential Equations, 2000 (2000), pp. 1-19.

[31] C. Tadjeran, M. M. Meerschaert and H. Scheffler, *A second-order accurate numerical approximation for the fractional diffusion equation*, J. Comput. Phys., 213(2006), pp. 205-213.

[32] C. Tadjeran, M. M. Meerschaert, *A second-order accurate numerical method for the two-dimensional fractional diffusion equation*, J. Comput. Phys., 220(2007), pp. 813-823.

[33] L. N. Trefethen, *Spectral Methods in MATLAB*, SIAM, Philadelphia, 2000.

[34] J. B. Walsh, *An introduction to stochastic partial differential equations*. In École d'été de probabilités de Saint-Flour, XI-1984, volume 1180 of Lecture Notes in Math., 265-439. Springer, Berlin, 1986.

[35] Y. Yan, *Galerkin finite element methods for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal. 43 (2005), pp. 13631384.

[36] G. M. Zaslavsky and S. S. Abdullaev, *Scaling property and anomalous transport of particals inside the stochastic layer*, Phys. Rev. E, (1995), 3901-3910.

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